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A multiple-sensor method for control of structural vibration with spatial objectives

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Abstract

This paper proposes a method to control vibration of arbitrary structures with spatially weighted objectives. Multiple discrete structural sensors are distributed over a structure and a spatial interpolation method is used to obtain the estimate of vibration at any points over the structure. The method thus does not require a priori information about the dynamic model of the structure. From the vibration information provided by structural sensors, spatial signals can then be obtained, representing the spatially weighted vibration of the entire structure. A condensation procedure allows the reduction of the required number of control input channels for active control purposes. A numerical case study of a flexible plate demonstrated that the proposed method can be used for minimising spatial vibration or for achieving a desired spatial vibration profile of a structure.

1. Introduction

Research into structural vibration control still receives a significant interest among researchers, partly due to the growing use of light-weight and flexible structures in many engineering applications. One challenging aspect of this research is that flexible structures are distributed systems, in which vibration at one structural location is related to vibration at other locations. This implies that controlling vibration at a few locations do not necessarily correspond to controlling the rest of structure. It is thus important to consider the overall spatial nature of structural vibration if one wants to control the vibration effectively.

For active structural vibration control, numerous control methods have been proposed such as the ones that utilise direct vibration information from one or more structural sensors, using various direct feedback control methods [1–3]. Many researchers have also proposed various spatial sensors for sensing vibration from a structure, using discrete or continuous sensors. The use of continuous vibration sensors such as shaped piezoelectric films [4] in particular polyvinylidene fluoride (PVDF) films, has been investigated by many researchers. The sensor can be shaped for measuring particular vibration or sound radiation characteristics that are of interest. For instance, such sensors have been used for controlling structural vibration [5] or sound radiation from vibrating structures using spatial sensors with porous electrodes [6], narrow strip sensors [7],

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transformed modal velocity sensors [8], volume displacement sensors [9] or quadratically weighted strain integrating sensors [10,11]. In these cases, accurate a priori information, such as the mode shapes and boundary conditions, may be needed to obtain precise sensor shapes for avoiding bias in the sensing. For some sensors, special electrodes also need to be manufactured, particularly when the sensors cover most of the structure [6]. The work presented here, however, attempts to develop spatial sensors (using discrete sensors) which do not depend on accurate a priori structural information that is generally needed if continuous sensors are used. This means that such spatial sensors can readily be used for structures with different vibration characteristics and boundary conditions.

The use of multiple discrete sensors in structural vibration control has also been common. Such sensors can be distributed over the structure to obtain a global vibration reduction, for instance by developing control systems that minimise error signals obtained from the sensors [12]. Meirovitch et al. [13,14] also proposed the use of multiple sensors via independent modal space control method [15] by extracting modal coordinates from sensor measurements, thus creating a modal filter which requires a priori information about the structure's modal properties. Other filtering methods are used for structural acoustic control. Elliot and Johnson [16] and Burgan et al. [17] both used velocity information from multiple discrete sensors to, respectively, create elemental radiators and an array of monopole sources for predicting the sound radiation power.

The work mentioned previously mainly deals with the control of some types of global structural vibration or sound radiation. In some cases, however, it is beneficial to control vibration at specific structural regions more than other regions. Other regions might be more prone to structural vibration or the vibration at certain regions might be more detrimental to the structural performance. For controlling vibration in certain spatial structural regions, some researchers have proposed the optimal $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control with spatially weighted objectives [18–20]. However, the control methods still rely on dynamic models of structures, which may not always be available in some applications.

As a result, the work in this paper proposes a different method for controlling spatial vibration using multiple structural sensors which allows specific structural regions to be continuously weighted more than the others. Spatial interpolations are used to estimate the spatial vibration profile of a structure. Although spatial interpolations have been used in other work [13,14,21,22], the previous work still requires a priori structural information such as structural mass/stiffness or modal properties. In contrast, the proposed method only uses vibration information directly from sensors without requiring a priori mass/stiffness or modal information. The method can thus provide a practical means of controlling structural vibration since it is less dependent on a priori structural information.

An example of specific applications of the control methods is for controlling the near-field sound/noise radiation in a vehicle cabin. Since the near-field sound radiation is heavily correlated with the velocity distribution of the radiating panel, the control method can be used to minimise the velocity profile of the panel so that the noise radiation can be minimised. An application for controlling sound radiation in particular far-field regions can also be targeted using the method.

2. Vibration sensing approach via spatial signals

Consider an arbitrary flexible structure whose a priori information is not available except for its geometric shape (see Fig. 1). In this situation it is not desirable to use a model-based control since the dynamic model of the plant is unknown. Even when a model is available, changes in dynamics occurring during control implementation in some instances might lead to performance reduction or even instability to the system. Therefore, the objective of this work is to control the spatial vibration of an arbitrary structure whose dynamic model is practically unknown. The spatial vibration is of particular interest here since it can be desirable to consider the vibration of the entire structure, and not just some points or regions on the structure.

2.1. Vibration profile estimation from a vibrating structure

To control spatial vibration of a structure effectively, it is necessary to estimate the structural vibration profile. The vibration profile of a structure can be estimated via spatial interpolation functions, using a method

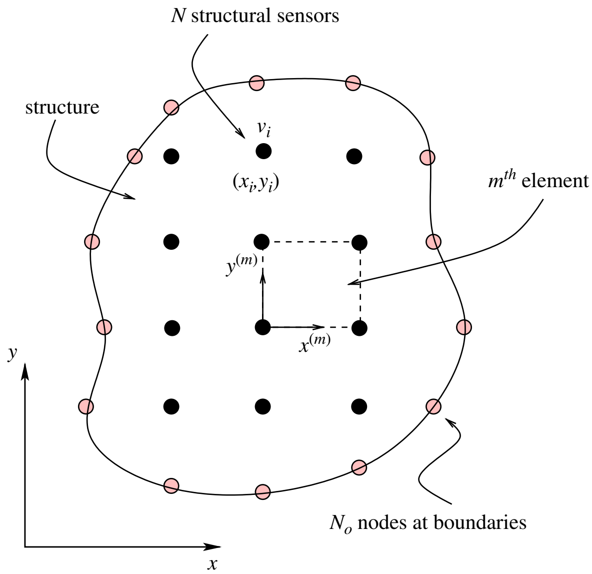

Fig. 1. A structure with N structural discrete sensors with N_o nodes at structural boundaries. Vibration measured at a single sensor at location (x_i, y_i) is v_i and the mth element, whose local coordinates are $(x^{(m)}, y^{(m)})$, are constructed from four nodes.

similar to the finite element method commonly used for numerical analysis of structures (see Refs. [23,24] for example). The principle of using spatial interpolations have also been discussed in other work such as in Refs. [13,21]. However, the work presented here also utilises information from structural boundary conditions for obtaining the vibration profile of the entire structure.

Consider the case where N structural discrete sensors are distributed over the structure, in which the ith sensor measures vibration v_i at a particular location (x_i, y_i), considering a two-dimensional system (see Fig. 1). Vibration information from N sensors can then be spatially interpolated from 'elements' generated by N nodes to generate a spatial vibration profile. In addition to vibration information from structural sensors, knowledge of structural boundary conditions can also improve the vibration profiling on regions close to structural boundaries. In the case where parts of structural boundaries have minimal vibrations, additional nodes/points can be included to provide additional 'elements' near the boundaries. In the following derivations, it is assumed that there are N_o locations/nodes/points at structural boundaries where vibrations are expected to be practically minimal.

Suppose there are M number of elements constructed from $N + N_o$ nodes over the structure, where N and N_o nodes are contributed by N structural sensors and N_o nodes at boundaries, respectively. Consider the mth element, whose local coordinates are $(x^{(m)}, y^{(m)})$ as shown in Fig. 1. In this case, the vibration level $\mathbf{v}_{xy}^{(m)}$ at any location $(x^{(m)}, y^{(m)})$ can be described by sensor measurements at l associated sensors for this particular mth element (for example, $l = 4$ in Fig. 1). In its most general form, $\mathbf{v}_{xy}^{(m)} \in \mathbf{R}^{k\times 1}$ will be a vector with k continuous vibration signals in terms of displacement, velocity, strain, or other vibration measures that can be derived from sensor signals $\mathbf{v}$:

$$\mathbf{v}_{xy}^{(m)}\left(x^{(m)}, y^{(m)}, t\right) = \mathbf{H}\left(x^{(m)}, y^{(m)}\right) \mathbf{v}^{(m)}(t), \tag{1}$$

where $\mathbf{v}^{(m)}$ consists of a group of l sensor measurements v_i associated with the mth element, and $\mathbf{H}(x^{(m)}, y^{(m)})$ is a $k \times l$ interpolation function matrix. Each set of local coordinates for the mth element can be transformed to the global coordinates (x, y) by using a linear transformation matrix $\mathbf{A}^{(m)} \in \mathbf{R}^{l\times(N+N_o)}$, so the vibration signals in global coordinates $\mathbf{v} \in \mathbf{R}^{(N+N_o)}$ can be related to the local coordinates:

$$\mathbf{v}^{(m)}(t) = \mathbf{A}^{(m)}\mathbf{v}(t). \tag{2}$$

Incorporating this coordinate transformation matrix for each element and substituting it into Eq. (1), the vibration at any point over the structure can be estimated with

$$\begin{aligned}\mathbf{v}_{xy}(x,y,t) &= \mathbf{H}\left(x^{(m)},y^{(m)}\right)\mathbf{A}^{(m)}\mathbf{v}(t)\\ &= \mathbf{M}(x,y)\mathbf{v},\end{aligned} \tag{3}$$

where $\mathbf{v}$ consists of sensor signals v_i and the nodes at boundaries with minimal vibrations. From Eq. (3), the structural vibration profile can be estimated from sensor measurements over the structure. Having estimated the vibration profile, the next task is to extract 'spatial signals' from the sensor measurements for active control purpose as discussed in the followings.

2.2. Choice of spatial interpolation functions

It is noted in Eq. (1), that the vibration profile is estimated by using spatial interpolation functions. Consequently, the control performance will be affected by the accuracy of the estimation. The choice of these functions depends on the types of structural sensors used, e.g. translational or angular vibration sensors. The lowest-order interpolation function is a linear function where the vibration profile between two structural sensors is linearly interpolated as shown in Fig. 2(a) for a one-dimensional case. Here, vibration levels are measured at the two ends of each structural element, which are used to interpolate the vibration level in between.

It is clear that by increasing the number of sensors distributed across the structure, a more accurate vibration profile interpolation can be achieved. However, there may be cases where it is not practically feasible to place sensors at such locations. For example, structural regions that are difficult to access physically or regions that requires 'clean' surfaces for aesthetic or performance reasons. In this case, multi-axis vibration sensors can be used to obtain an accurate interpolation without requiring sensors to be placed at some locations. For this purpose, micro electro mechanical system (MEMS) sensor technology has been widely available for years with multi-axis vibration sensing capability. These MEMS sensors such the ones from Analog Devices are continuously being improved which results in decreasing sensor size, operational power and price.

MEMS sensors can be utilised for measuring a range of translational and angular vibrations that can be used in conjunction with a higher-order polynomial for interpolating the vibration profile. Fig. 2(b) illustrates a higher-order interpolation function used when translational and angular vibration sensors are located at the two ends of the element, allowing a more accurate interpolation of the vibration profile than that of the linear interpolation.

The spatial interpolation functions need to be chosen such that at the location of each sensor, the estimated vibration profile $\mathbf{v}_{xy}^{(m)}$ in Eq. (1) must converge to the vibration level measured by that particular sensor. Suppose that a particular sensor is located at $(x_1^{(m)}, y_1^{(m)})$ at the mth element, then the interpolation functions

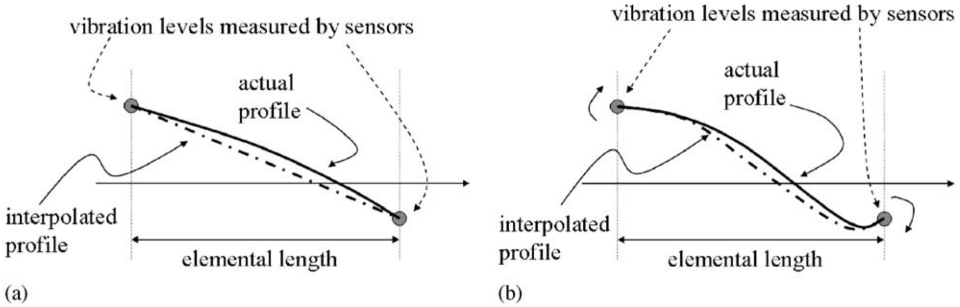

Fig. 2. Interpolation functions for estimating the vibration profile of a structure: (a) linear interpolation; (b) cubic interpolation.

must be chosen such that:

$$\begin{aligned}\mathbf{v}_{xy}^{(m)}(x_1^{(m)}, y_1^{(m)}, t) &= \mathbf{H}(x_1^{(m)}, y_1^{(m)})\mathbf{v}^{(m)}(t) \\ &= \mathbf{v}_1^{(m)}(t),\end{aligned} \tag{4}$$

where $\mathbf{v}_1^{(m)}$ consists of the vibration levels measured by sensors located at $(x_1^{(m)}, y_1^{(m)})$.

A particular approach that can be used for interpolations is to assume separable interpolation function with respect to x and y coordinates:

$$\mathbf{H}\left(x^{(m)}, y^{(m)}\right) = \mathbf{H}_x\left(x^{(m)}\right)\mathbf{H}_y\left(y^{(m)}\right), \tag{5}$$

where $\mathbf{H}_x\left(x^{(m)}\right)$ and $\mathbf{H}_y\left(y^{(m)}\right)$ are the spatial interpolation matrices for x and y coordinates, respectively.

For example, suppose that one is interested to a scalar velocity profile over a structure, i.e. $v_{xy}^{(m)} \in \mathbf{R}$ (Eq. (1)). Consider the linear interpolation with respect to one of the coordinates, similar to the one shown in Fig. 2. When a transverse velocity sensor is used at each node, a linear interpolation function can be used so that the velocity profile at $y^{(m)} = 0$ is

$$v_{xy}^{(m)}\left(x^{(m)}, 0, t\right) = \begin{bmatrix} 1 - \dfrac{x^{(m)}}{h_x^{(m)}} & \dfrac{x^{(m)}}{h_x^{(m)}} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}, \tag{6}$$

where v_1 and v_2 are the velocity levels measured at the two sensors, while $x^{(m)}$ and $h_x^{(m)}$ are the elemental x coordinate and length, respectively.

When a transverse velocity sensor and an angular velocity sensor are used for each node, the Hermite cubic interpolation function can be used instead so that the velocity profile at $y^{(m)} = 0$ is now:

$$v_{xy}^{(m)}\left(x^{(m)}, 0, t\right) = \begin{bmatrix} 1 - 3\left(\dfrac{x^{(m)}}{h_x^{(m)}}\right)^2 + 2\left(\dfrac{x^{(m)}}{h_x^{(m)}}\right)^3 \\ h_x^{(m)}\left\{\left(\dfrac{x^{(m)}}{h_x^{(m)}}\right) - 2\left(\dfrac{x^{(m)}}{h_x^{(m)}}\right)^2 + \left(\dfrac{x^{(m)}}{h_x^{(m)}}\right)^3\right\} \\ 3\left(\dfrac{x^{(m)}}{h_x^{(m)}}\right)^2 - 2\left(\dfrac{x^{(m)}}{h_x^{(m)}}\right)^3 \\ h_x^{(m)}\left\{-\left(\dfrac{x^{(m)}}{h_x^{(m)}}\right)^2 + \left(\dfrac{x^{(m)}}{h_x^{(m)}}\right)^3\right\} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ v_4(t) \end{bmatrix}, \tag{7}$$

where v_1 and v_3 are the linear velocity levels measured at the two linear sensors, while v_2 and v_4 are the angular velocity levels measured at the two angular sensors.

2.3. Spatial signals

Consider the case when controlling vibration at some regions over a structure are more desirable than other regions. In this case, it can be useful to consider a spatially weighted vibration control approach so important regions can be emphasised accordingly. Further, in some applications, it might also be useful to force the vibration profile of an arbitrary structure to a desired spatial vibration profile. For example, a desired spatial vibration profile of a structure might be controlled to improve its aerodynamic performance. For this purpose, it is convenient to introduce 'spatial signals' to obtain this desired spatially weighted objectives. These spatial signals can be incorporated with standard active control methods such as various adaptive control methods [12] for structural vibration control.

Next, consider a general case in which a desired spatial vibration profile is described by $\mathbf{f}_{xy}(x, y, t)$. A real symmetric spatial weighting matrix $\mathbf{W}(x, y)$ is introduced to emphasise structural regions that are more important to be controlled. Here, $\mathbf{W}(x, y) \geqslant 0$ for all locations of points (x, y) in a structural region of interest R.

In this work, the following spatial error function $\mathbf{e}(x, y, t)$ is introduced:

$$\mathbf{e}(x, y, t) = \mathbf{v}_{xy}(x, y, t) - \mathbf{f}_{xy}(x, y, t), \tag{8}$$

where $\mathbf{v}_{xy}$ is the spatial vibration profile of the structure.

The following objective/cost function $J(t)$ is now proposed:

$$\begin{aligned}
J(t) &= \int_R \mathbf{e}(x, y, t)^{\mathrm{T}} \mathbf{W}(x, y) \mathbf{e}(x, y, t)\, \mathrm{d}R \\
&= \int_R \left(\mathbf{v}_{xy}(x, y, t) - \mathbf{f}_{xy}(x, y, t)\right)^{\mathrm{T}} \mathbf{W}(x, y)(\mathbf{v}_{xy}(x, y, t) - \mathbf{f}_{xy}(x, y, t))\, \mathrm{d}R \\
&= \mathbf{v}(t)^{\mathrm{T}} \int_R \mathbf{M}(x, y)^{\mathrm{T}} \mathbf{W}(x, y) \mathbf{M}(x, y)\, \mathrm{d}R \mathbf{v}(t) - \mathbf{v}(t)^{\mathrm{T}} \int_R \mathbf{M}(x, y)^{\mathrm{T}} \\
&\quad \times \mathbf{W}(x, y) \mathbf{f}(x, y, t)\, \mathrm{d}R - \int_R \mathbf{f}(x, y, t)^{\mathrm{T}} \mathbf{W}(x, y) \mathbf{M}(x, y)\, \mathrm{d}R \mathbf{v}(t) \\
&\quad + \int_R \mathbf{f}(x, y, t)^{\mathrm{T}} \mathbf{W}(x, y) \mathbf{f}(x, y, t)\, \mathrm{d}R,
\end{aligned} \tag{9}$$

where $(F)^{\mathrm{T}}$ represents the transpose of a matrix F and R is again the region of the structure. It can be shown that the cost function can be represented as

$$J(t) = (\mathbf{v}(t) - \mathbf{v}^o(t))^{\mathrm{T}} \mathbf{A}(\mathbf{v}(t) - \mathbf{v}^o(t)) + c(t), \tag{10}$$

with

$$\mathbf{v}^o(t) = \mathbf{A}^{-1} \mathbf{b}(t) \tag{11}$$

and

$$\begin{aligned}
\mathbf{A} &= \int_R \mathbf{M}(x, y)^{\mathrm{T}} \mathbf{W}(x, y) \mathbf{M}(x, y)\, \mathrm{d}R, \\
\mathbf{b}(t) &= \int_R \mathbf{M}(x, y)^{\mathrm{T}} \mathbf{W}(x, y) \mathbf{f}(x, y, t)\, \mathrm{d}R, \\
c(t) &= \int_R \mathbf{f}(x, y, t)^{\mathrm{T}} \mathbf{W}(x, y) \mathbf{f}(x, y, t)\, \mathrm{d}R - \mathbf{v}^o(t)^{\mathrm{T}} \mathbf{b}(t).
\end{aligned} \tag{12}$$

Note that $\mathbf{W}(x, y)$ is Hermitian, so $\mathbf{A}$ can be shown to be Hermitian. In general, the term $\mathbf{M}(x, y)^{\mathrm{T}} \mathbf{W}(x, y) \mathbf{M}(x, y) \geqslant 0$ for every point (x, y) in region R. Since the term is integrated over the entire region R, it can be shown that $\mathbf{A}$ is typically positive definite, i.e. $\mathbf{A} > 0$. This implies that $\mathbf{A}$ is non-singular and $\mathbf{v}^o(t)$ can be obtained from Eq. (11).

Now, matrix $\mathbf{A}$ is real and symmetric since it is Hermitian, and it can be decomposed into:

$$\mathbf{A} = \mathbf{\Omega}_{\mathrm{sp}}^{\mathrm{T}} \mathbf{\Omega}_{\mathrm{sp}}. \tag{13}$$

Since there are N_o nodes that are constrained at structural boundaries, $\mathbf{A}$ and $\mathbf{b}(t)$ can be condensed by removing the appropriate rows and columns that correspond to the associated nodes. This condensation reduces the dimensions of $\mathbf{A}$ and $\mathbf{b}(t)$ from $N + N_o$ to N.

Incorporating Eq. (13), the cost function J in Eq. (10) can then be simply represented as

$$\begin{aligned}
J(t) &= (\mathbf{v}(t) - \mathbf{v}^o(t))^{\mathrm{T}} \mathbf{\Omega}_{\mathrm{sp}}^{\mathrm{T}} \mathbf{\Omega}_{\mathrm{sp}} (\mathbf{v}(t) - \mathbf{v}^o(t)) + c(t) \\
&= (\mathbf{v}_{\mathrm{sp}}(t) - \mathbf{v}_{\mathrm{sp}}^o(t))^{\mathrm{T}} (\mathbf{v}_{\mathrm{sp}}(t) - \mathbf{v}_{\mathrm{sp}}^o(t)) + c(t) \\
&= \tilde{J}(t) + c(t),
\end{aligned} \tag{14}$$

where $\tilde{J}$ is as defined above and

$$\mathbf{v}_{\text{sp}}(t) = \mathbf{\Omega}_{\text{sp}}\mathbf{v}(t),$$

$$\mathbf{v}^{o}_{\text{sp}}(t) = \mathbf{\Omega}_{\text{sp}}\mathbf{v}^{o}(t) = \mathbf{\Omega}_{\text{sp}}\mathbf{A}^{-1}\mathbf{b}(t). \tag{15}$$

Hence, $\mathbf{v}_{\text{sp}}$ is a spatial signal representing the spatially weighted vibration over the entire structure, whereas $\mathbf{v}^{o}_{\text{sp}}$ is a spatial signal representing the desired spatial vibration profile. It can be observed that $c(t)$ in Eq. (14) is not affected by the level of structural vibration so it is sufficient to use cost function $\tilde{J}$ instead (as described in Eq. (14)) in cost function minimisation for active control.

The important implication of describing the cost function via spatial signals $\mathbf{v}_{\text{sp}}$ is that a simple sensing procedure can be obtained by filtering the sensor signals with $\mathbf{\Omega}_{\text{sp}}$ as in Eq. (15). This method essentially filters the signals from structural sensors to produce a spatial signal that takes into account a spatially weighted objective. This spatial signal $\mathbf{v}_{\text{sp}}$ can be subtracted by the desired spatial vibration profile $\mathbf{v}^{o}_{\text{sp}}$ (see Eq. (14)), and can be regarded as the error signals whose energy $\tilde{J}$ is to be minimised. A standard active control algorithm can then be conveniently implemented into the system by using these spatial error signals.

2.4. Choice of spatial weighting functions

As previously mentioned, a real symmetric spatial weighting matrix $\mathbf{W}(x, y)$ can be used to emphasise structural regions whose vibrations are more important to be controlled. The weighting matrix can be tailored according to the performance requirement. For instance, consider controlling vibration displacement and strain of a flexible structure. There may be a region in the structure where it is important to minimise displacement vibration for functional purposes such as to improve the aerodynamic or tracking performances. On the other hand, other regions may require more emphasis in minimising the strain to improve its fatigue life. These performance objectives can be conveniently taken into account by spatial weighting matrix $\mathbf{W}(x, y)$. A diagonal matrix, for example, can be used where its diagonal term represent a spatial weighting function for each vibration displacement or strain:

$$\mathbf{W}(x, y) = \begin{bmatrix} w_1(x, y) & 0 \\ 0 & w_2(x, y) \end{bmatrix}, \tag{16}$$

where $w_1(x, y), w_2(x, y) > 0\, \forall\, (x, y) \in R$, representing scalar spatial weighting functions for vibration displacement or strain, respectively, in this example. Polynomial functions in x and y can easily be used for generating the required spatial weighting function. A full $\mathbf{W}(x, y)$ matrix can also be used which includes the cross-weighting between the displacement and strain objectives.

Another example is where some structural regions tend to radiate noise more. These regions can be specifically targeted by choosing a spatial weighting function that emphasise these regions more than other regions.

Another advantage is that the obtained filter matrix $\mathbf{\Omega}_{\text{sp}}$ in Eq. (15) is based on the spatial weighting matrix used. This implies that vibration objectives can be changed during active control operation by changing the filter matrix used without changing or re-locating the sensors. This makes it convenient to achieve a range of different control objectives during the control operation without 're-touching' the structure and sensors.

2.5. Calculation of filter matrix $\mathbf{\Omega}_{\text{sp}}$

In order to calculate the filter matrix $\mathbf{\Omega}_{\text{sp}}$ in Eq. (13), the eigenvalue decomposition approach is used as follows:

$$\begin{aligned} \mathbf{A} &= \mathbf{\Omega}^{\text{T}}_{\text{sp}}\mathbf{\Omega}_{\text{sp}} \\ &= \mathbf{U}\mathbf{V}\mathbf{U}^{\text{T}}, \end{aligned} \tag{17}$$

where $\mathbf{V}$ is a diagonal matrix containing positive real valued eigenvalues; $\mathbf{U}$ is the matrix containing the associated eigenvectors and $\mathbf{U}$ is unitary, i.e. $\mathbf{U}\mathbf{U}^{\mathrm{T}} = \mathbf{I}$. Thus,

$$\mathbf{\Omega}_{\mathrm{sp}} = \mathbf{V}^{1/2}\mathbf{U}^{\mathrm{T}} \tag{18}$$

and the spatial signals to be used for control purpose can be obtained from sensor measurements, $\mathbf{v}_{\mathrm{sp}} = \mathbf{\Omega}_{\mathrm{sp}}\mathbf{v}$ as in Eq. (15).

2.6. Condensation of spatial signals

There is a question on how many sensors would be needed to sufficiently estimate the vibration profile of a structure. When linear interpolation functions are used, a rule of thumb is to use about 5–6 nodes per spatial wavelength in order to obtain reasonably accurate profile estimation, such as to avoid spatial aliasing [25]. These nodes might also include the nodes at the boundaries, which implies that less number of actual sensors would be required. Furthermore, when higher-order interpolation functions are used in conjunction with combinations of translational and angular sensors, the number of nodes required per spatial wavelength would considerably lower. However, the proposed method leads to as many spatial signals as structural sensors used, possibly leading to the need for a computationally expensive active control system with large multi-channel control inputs. Obviously in practice, it would be desirable to reduce the number of control inputs used for active control. The following method allows the condensation of spatial signals so less number of control inputs can be used.

The number of spatial signals can be reduced by considering the eigenvalue decomposition of $\mathbf{\Omega}_{\mathrm{sp}}^{\mathrm{T}}\mathbf{\Omega}_{\mathrm{sp}}$:

$$\mathbf{\Omega}_{\mathrm{sp}}^{\mathrm{T}}\mathbf{\Omega}_{\mathrm{sp}} = \sum_{i=1}^{N} \lambda_i \mathbf{u}_i \mathbf{u}_i^{\mathrm{T}}, \tag{19}$$

where N is again the number of sensors used.

The importance of each eigenmode can now be gauged by the size of its corresponding eigenvalue. Therefore, a condensation of the spatial signal can be obtained by including the more important eigenmodes:

$$\tilde{\mathbf{\Omega}}_{\mathrm{sp}}^{\mathrm{T}}\tilde{\mathbf{\Omega}}_{\mathrm{sp}} = \sum_{i=1}^{N_r} \lambda_i \mathbf{u}_i \mathbf{u}_i^{\mathrm{T}}, \tag{20}$$

where $N_r < N$, which implies that only the contribution of N_r largest eigenvalues are considered in the calculation of the estimated matrix $\tilde{\mathbf{\Omega}}_{\mathrm{sp}}$. Hence, through the signal condensation, less numerical computation and less number of control input channels are required for active control purposes.

The summary of the method for obtaining the spatial signal is shown in Fig. 3. The sensor signals are filtered to produce a spatial signal that can be used in conjunction with an active control algorithm and control

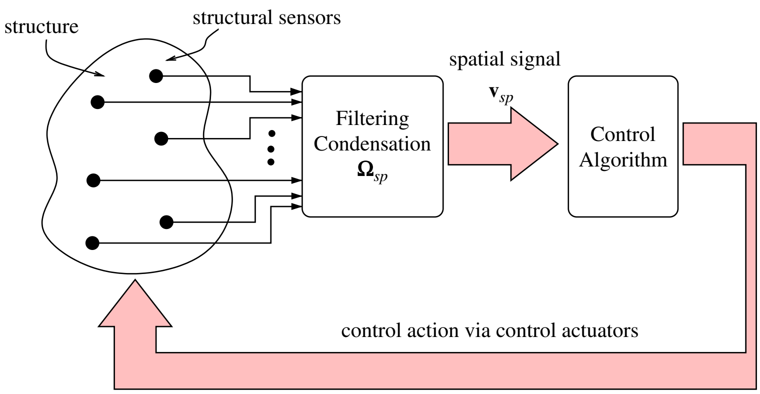

Fig. 3. A general approach for sensing and control using spatial signals.

actuators to control vibration of a structure. Various types of control actuators such as electrodynamic shakers and piezoelectric patches can be used for control. As with other control methods, for an efficient control performance, control actuators need to be located at locations where they have a relatively high level of controllability for vibration modes of interest.

In general, a finite number of discrete sensors are used and control actuators are not collocated to the sensors, which means that control and observation spillover problem [26,27] may also occur in the proposed spatial control method. The control spillover may affect the control performance since the control actuators can excite vibrations at frequencies outside the control bandwidth, which means that the desired spatial control objective cannot be optimally minimised. The observation spillover can occur because the discrete sensors used can detect vibrations at frequencies outside the control bandwidth, which may affect the control performance and even destabilise the control system. Since the sensors are distributed over the structure and the observability level of each vibration mode varies with locations, the level of observation spillover will vary for each sensor depending on where the sensor is located. The spillover problem is particularly important for broadband control and it should not be a real concern for tonal control. For broadband control, the effect of spillover can be minimised by low-pass filtering the spatial signals and control input signals to reduce the effect of observing and exciting higher frequency vibration modes outside the bandwidth of interest.

In addition to the standard spillover problem, the spatial aliasing [25] can be detrimental to the performance and stability of broadband spatial control. Vibrations at higher frequency modes can be aliased into those at lower frequency modes because the number of sensors is not sufficient to provide an accurate spatial profile estimation at higher frequencies. For tonal control, an inaccurate spatial estimation would lead to an inaccurate spatial objective function that can compromise the control performance and even affect the control stability. It is thus important to minimise the spatial aliasing by using a sufficient number of sensors for sensing vibration modes within the control bandwidth. Although sufficient number of discrete sensors are required, less expensive sensors such as small PVDF patches can also be used for the same control purpose. Low-pass filtering of spatial and control input signals would also be useful to minimise the spatial aliasing problem.

3. Optimal spatial control for tonal disturbances

In this section, optimal spatial control for tonal disturbances is considered to demonstrate the proposed sensing approach for active control. The tonal vibration control case is chosen since it demonstrates clearly the effect of spatial control approach in modifying the spatial vibration profile of a structure. A feedforward case is considered in which a reference signal for the disturbance is available.

Suppose there are primary disturbances $\mathbf{d}$ at frequency ω_o that cause structural vibration. Secondary disturbances are required to spatially control the vibration, in which control signals are denoted by $\mathbf{u}$. Then the vibration levels $\mathbf{v}$ at N multiple sensors attached to the structure can be described by

$$\mathbf{v}(j\omega_o) = \mathbf{G}_{vd}(j\omega_o)\mathbf{d}(j\omega_o) + \mathbf{G}_{vu}(j\omega_o)\mathbf{u}(j\omega_o), \tag{21}$$

where $\mathbf{G}_{pq}$ represents the transfer matrix from $\mathbf{q}$ to $\mathbf{p}$.

The cost function $\tilde{J}$, as in Eq. (14):

$$\tilde{J} = (\mathbf{v}_{\mathrm{sp}} - \mathbf{v}^o_{\mathrm{sp}})^H(\mathbf{v}_{\mathrm{sp}} - \mathbf{v}^o_{\mathrm{sp}}), \tag{22}$$

where $(F)^H$ is the Hermitian transpose of matrix F and $(j\omega_o)$ term has been omitted for the sake of brevity.

From Eq. (15), the spatial signal is

$$\mathbf{v}_{\mathrm{sp}} = \mathbf{\Omega}_{\mathrm{sp}}\mathbf{G}_{vd}\mathbf{d} + \mathbf{\Omega}_{\mathrm{sp}}\mathbf{G}_{vu}\mathbf{u},$$

$$\mathbf{v}^o_{\mathrm{sp}} = \mathbf{\Omega}_{\mathrm{sp}}\,\mathbf{v}^o = \mathbf{\Omega}_{\mathrm{sp}}\mathbf{A}^{-1}\mathbf{b}, \tag{23}$$

where

$$\mathbf{A} = \int_R \mathbf{M}(x, y)^{\mathrm{T}} \mathbf{W}(x, y) \mathbf{M}(x, y) \, \mathrm{d}R,$$

$$\mathbf{b} = \int_R \mathbf{M}(x, y)^{\mathrm{T}} \mathbf{W}(x, y) \mathbf{f}(x, y) \, \mathrm{d}R. \tag{24}$$

Note that the condensed spatial signal based on the estimated $\tilde{\mathbf{\Omega}}_{\mathrm{sp}}$ can also be used by replacing $\mathbf{\Omega}_{\mathrm{sp}}$ with $\tilde{\mathbf{\Omega}}_{\mathrm{sp}}$.

Substituting Eq. (23) into Eq. (22) and using a quadratic minimisation approach [12], the optimal control input $\mathbf{u}_{\mathrm{opt}}$ can be obtained:

$$\mathbf{u}_{\mathrm{opt}} = -(\mathbf{G}_{vu}^H \mathbf{\Omega}_{\mathrm{sp}}^{\mathrm{T}} \mathbf{\Omega}_{\mathrm{sp}} \mathbf{G}_{vu})^{-1} \times \mathbf{G}_{vu}^H \mathbf{\Omega}_{\mathrm{sp}}^{\mathrm{T}} (\mathbf{\Omega}_{\mathrm{sp}} \mathbf{G}_{vd} \mathbf{d} - \mathbf{v}_{\mathrm{sp}}^o). \tag{25}$$

Note that the transpose of $\mathbf{\Omega}_{\mathrm{sp}}$ is used instead of its Hermitian transpose since $\mathbf{\Omega}_{\mathrm{sp}}$ is a real matrix. Hence, the optimal control signal depends on the filter matrix $\mathbf{\Omega}_{\mathrm{sp}}$ and the desired spatial vibration profile $\mathbf{v}_{\mathrm{sp}}^o$.

The minimum cost function $\tilde{J}_{\mathrm{min}}$ is

$$\tilde{J}_{\mathrm{min}} = J_o - (\mathbf{d}^H \mathbf{G}_{vd}^H \mathbf{\Omega}_{\mathrm{sp}}^{\mathrm{T}} - \mathbf{v}_{\mathrm{sp}}^{oH}) \mathbf{\Omega}_{\mathrm{sp}} \mathbf{G}_{vu} \mathbf{u}_{\mathrm{opt}}, \tag{26}$$

where $\tilde{J}_o$ is the initial cost function before control given by

$$\tilde{J}_o = \mathbf{d}^H \mathbf{G}_{vd}^H \mathbf{\Omega}_{\mathrm{sp}}^{\mathrm{T}} \mathbf{\Omega}_{\mathrm{sp}} \mathbf{G}_{vd} \mathbf{d} - \mathbf{d}^H \mathbf{G}_{vd}^H \mathbf{\Omega}_{\mathrm{sp}}^{\mathrm{T}} \mathbf{v}_{\mathrm{sp}}^o - \mathbf{v}_{\mathrm{sp}}^{oH} \mathbf{\Omega}_{\mathrm{sp}} \mathbf{G}_{vd} \mathbf{d} + \mathbf{v}_{\mathrm{sp}}^{oH} \mathbf{v}_{\mathrm{sp}}^o. \tag{27}$$

In the following section, the implementation of the proposed sensing and control approach is investigated on a flexible plate structure.

4. A numerical case study: control of a flexible plate

A simply-supported aluminium plate has been used for this case study since its dynamic behaviour is widely known. There are $N = 25$ velocity sensors distributed evenly across the plate with the arrangement of sensors similar to the one shown in Fig. 4. Point forces are used as disturbance and control sources. The dynamic model of the plate is obtained using the modal analysis method [28] by taking into account the first 15 modes of the plate. In this case study, structural control of only up to the first four modes (up to about 400 Hz) is considered, hence 15 modes of up to 1153.5 Hz are deemed to be sufficient for the model of the plate. Properties of the plate is shown in Table 1, while the resonance frequencies of the first four modes are shown in Table 2.

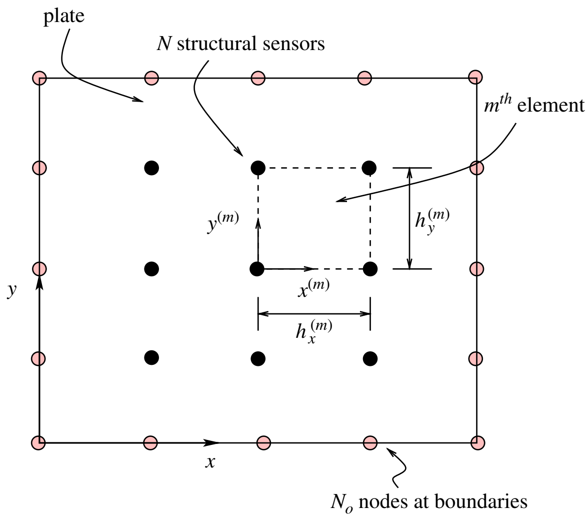

Fig. 4. A flexible plate with multiple sensors.

Table 1
Properties of the plate used in the numerical case study

Plate x-length	0.500 m
Plate y-length	0.400 m
Plate thickness	0.004 m
Plate Young's modulus	7.0×10^{10} N/m^2
Plate Poisson's ratio	0.30
Plate density	2750 kg/m^3

Table 2
Resonance frequencies of the first four modes of the plate

No.	Mode	Frequency (Hz)
1	(1,1)	98.3
2	(2,1)	213.4
3	(1,2)	278.2
4	(2,2)	393.3

Linear interpolation functions and rectangular elements are used to estimate the vibration profile of the plate. For each rectangular element, $l = 4$ sensors at all four corners are used as the nodes, whose dimension is $h_x^{(m)}$ and $h_y^{(m)}$ in $x^{(m)}$ and $y^{(m)}$ directions, respectively. The matrix of interpolation functions Eq. (1) is

$$\mathbf{H}(x^{(m)}, y^{(m)}) = \begin{bmatrix} \left\{1 - \left(\dfrac{x^{(m)}}{h_x^{(m)}}\right)\right\}\left\{1 - \left(\dfrac{y^{(m)}}{h_y^{(m)}}\right)\right\} \\ \left(\dfrac{x^{(m)}}{h_x^{(m)}}\right)\left\{1 - \left(\dfrac{y^{(m)}}{h_y^{(m)}}\right)\right\} \\ \left\{1 - \left(\dfrac{x^{(m)}}{h_x^{(m)}}\right)\right\}\left(\dfrac{y^{(m)}}{h_y^{(m)}}\right) \\ \left(\dfrac{x^{(m)}}{h_x^{(m)}}\right)\left(\dfrac{y^{(m)}}{h_y^{(m)}}\right) \end{bmatrix}^{\mathrm{T}}. \tag{28}$$

In this case study, a scalar continuous velocity signal v_{xy} is given by $v_{xy}(x, y, t) = \mathbf{M}(x, y)\mathbf{v}$ from Eq. (3) where $\mathbf{M}(x, y) = \mathbf{H}(x^{(m)}, y^{(m)})\mathbf{A}^{(m)}$ when a particular mth element is of interest.

A scalar spatial weighting function $W(x, y) > 0$ is chosen and $\mathbf{\Omega}_{\mathrm{sp}}$ can be obtained from Eq. (13) with the associated $\mathbf{A}^{(m)}$ after considering the contributions from all M elements used:

$$\mathbf{\Omega}_{\mathrm{sp}}^{\mathrm{T}}\mathbf{\Omega}_{\mathrm{sp}} = \sum_{m=1}^{M}\left\{\int_{R^{(m)}} \mathbf{A}^{(m)\mathrm{T}}\left(x^{(m)}, y^{(m)}\right)\mathbf{H}^{(m)\mathrm{T}}\left(x^{(m)}, y^{(m)}\right) W\left(x^{(m)}, y^{(m)}\right)\mathbf{H}^{(m)}\mathbf{A}^{(m)}\,\mathrm{d}R^{(m)}\right\}, \tag{29}$$

where $R^{(m)}$ represents the region of the mth element.

4.1. Control of spatially weighted vibration of a plate

The first numerical study considers the vibration minimisation of the plate. Point forces are used for a disturbance source and two control sources whose (x, y) locations are (0.313, 0.121 m) and (0.156, 0.121 m), (0.278, 0.191 m), respectively. A scalar spatial weighting function $W(x, y) > 0$ is chosen in which some regions receive larger weights than the others, as shown in Fig. 5. The desired vibration profile is set to $f_{xy}(x, y) = 0$ since only vibration minimisation is of interest here. Here, a condensed spatial signal is to be used for active

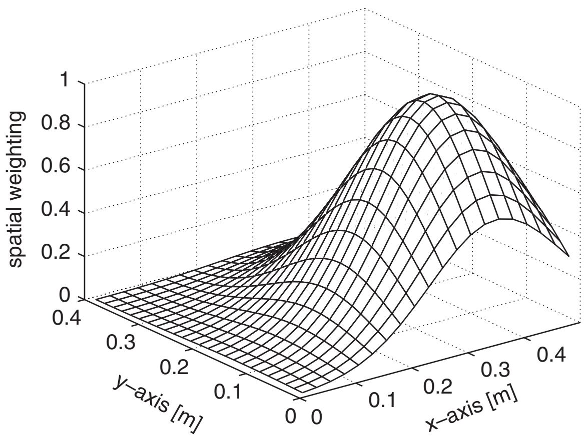

Fig. 5. The spatial weighting function, emphasising the region of the plate where vibration reductions are more important.

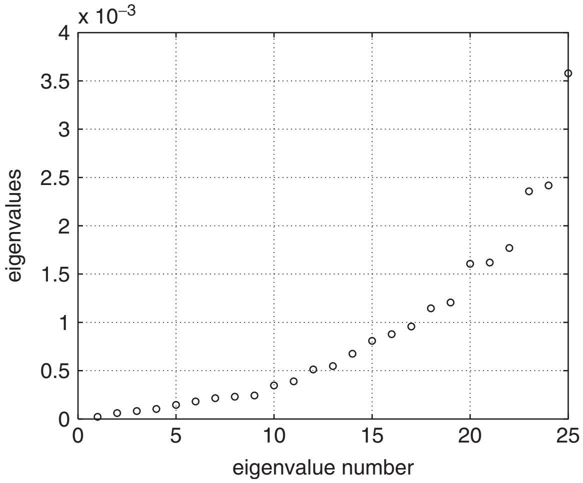

Fig. 6. The eigenvalues plot of $\mathbf{\Omega}_{sp}^{T}\mathbf{\Omega}_{sp}$ for vibration control of the plate. Six largest eigenvalues were used for condensing the size of the spatial filter.

control of the plate since a full number of spatial signals would require 25 control input channels. To do this, eigenvalues of $\mathbf{\Omega}_{sp}^{T}\mathbf{\Omega}_{sp}$ are obtained as shown in Fig. 6. It was decided to use the six largest eigenvalues to generate a condensed spatial signal and investigate the performance of active control based on the signal. After the condensation, only six control inputs would be required.

Now, the control problem is to find a control input that minimise the cost function $\tilde{J}$ in Eq. (22), where the optimal control $\mathbf{u}_{opt}$ can be obtained from Eq. (25). To demonstrate the control performance in reducing structural vibration, two vibration modes are chosen, i.e. modes (1,1) and (2,2). A tonal disturbance input is selected $d = \sin(2\pi f_1 t)$, where $f_1 = 98.3$ Hz is the natural frequency of mode (1,1). The results using a single control source are shown in Fig. 7 for controlling mode (1,1), where the control inputs for standard control and spatial control are $u = 1.1111\sin(2\pi f_1 t - 3.142)$ and $u = 1.1112\sin(2\pi f_1 t - 3.138)$, respectively. Figs. 7(b) and (c) compare the active control performances using standard and spatial control methods, respectively. The standard control method simply minimises the sum of the squared error signals of all N velocity sensors distributed over the plate. The results can be compared against the un-controlled vibration profile in Fig. 7(a). It can be seen that the spatial control method managed to reduce vibration more in the region where the spatial weighting was larger. Although the control inputs for both standard and spatial control methods are

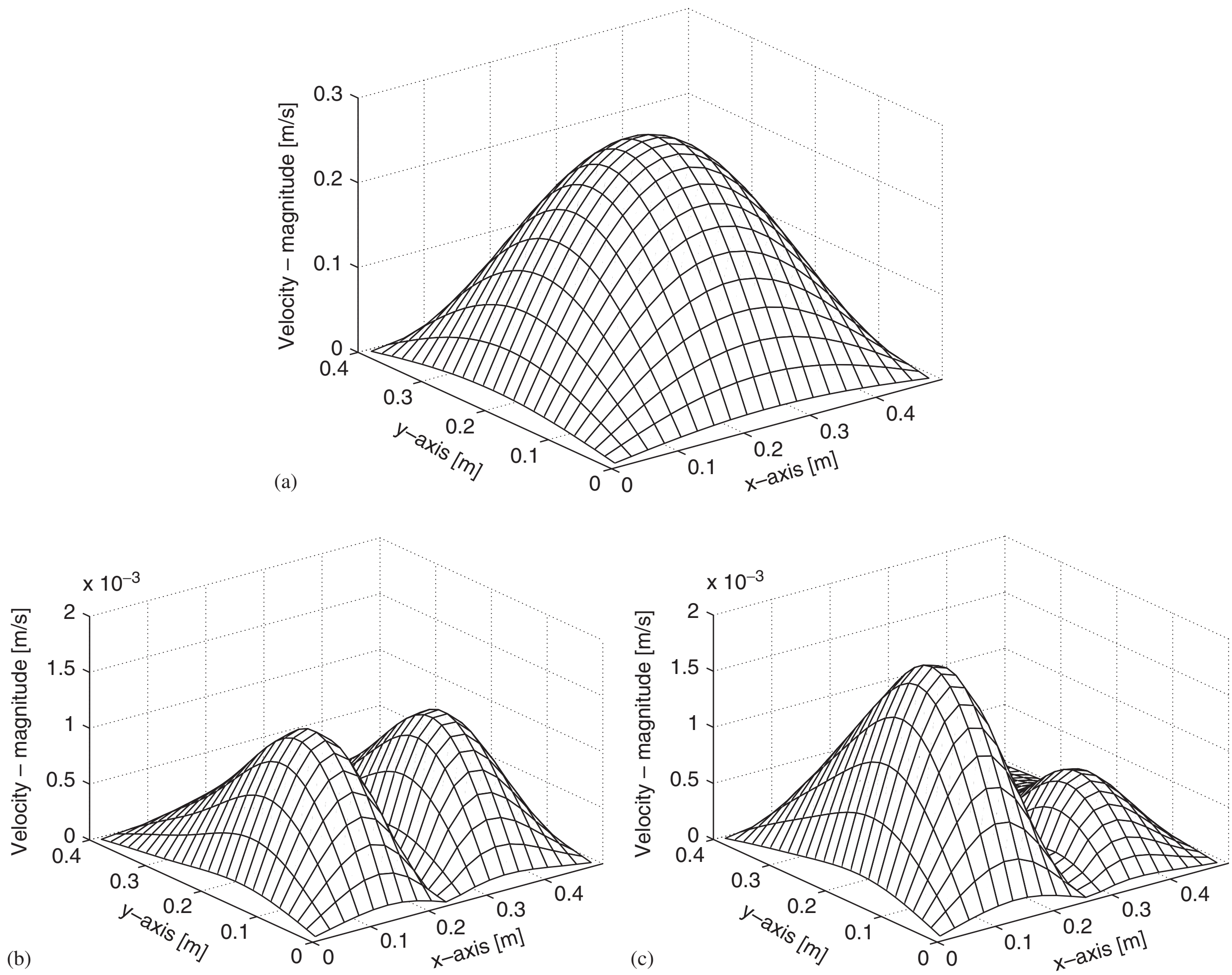

Fig. 7. Comparison of standard control and spatial control for vibration mode (1,1) using one control source: (a) no control—spatial vibration profile of mode (1,1); (b) standard control; (c) spatial control.

close to each other, a slight phase difference allows the spatial control to decrease the vibration level at the right-hand side region of the plate by increasing the vibration level at the left-hand side region, as compared to the standard control results. The use of spatial control method can be beneficial when vibrations on certain regions in the structure are more important than those at other regions.

It is also possible for the standard control method to use a different weight at each error signal to achieve a particular spatial weighting, which amounts to multiplying the error signals with a diagonal weighting matrix. However, the number of error signals cannot be reduced, which implies that the controller requires as many input channels as the number of sensors used. This is a significant advantage for the proposed method since less complex and more practical controller can be used.

In general, the proposed method can be seen as an extension of the standard control for structural vibration control which allows continuous spatial weighting to be used. The objective function defined over the structure also provides a natural expression of structural vibration. As explained in Section 2.2, when a combination of translational and angular vibration sensors are used, the proposed method allows the sensor signals to be combined explicitly to obtain the estimation of physical vibration profile, which is less obvious when a standard method is used.

Similar results can be seen in Fig. 8 for controlling mode $(2,2)$ where the spatial control method achieved greater vibration reductions at the regions where the spatial weighting was larger. The disturbance input used

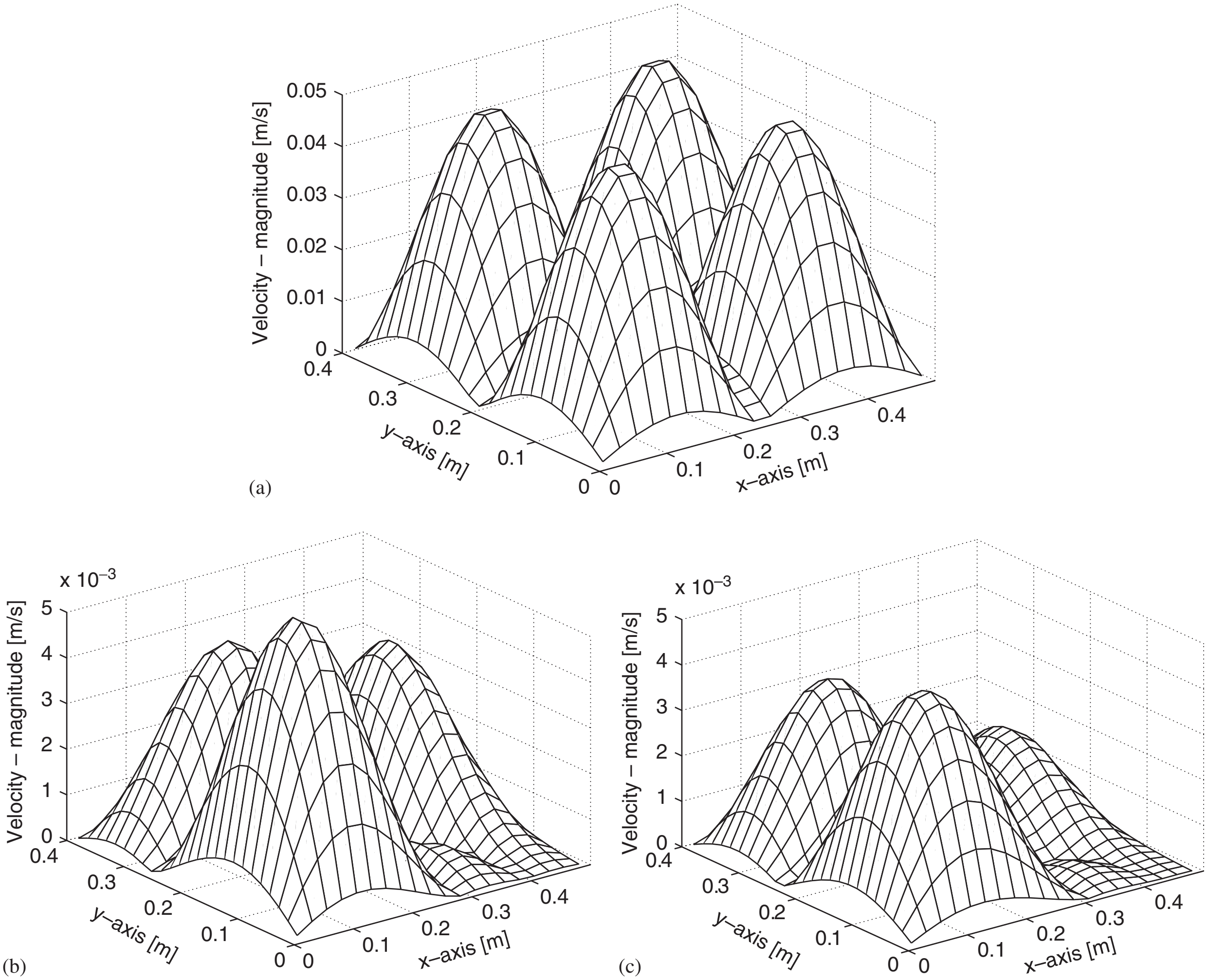

Fig. 8. Comparison of spatial control with one control and two control sources for vibration mode (2,2): (a) no control—spatial vibration profile of mode (2,2); (b) one control source; (c) two control sources.

is $d = \sin(2\pi f_4 t)$, where $f_4 = 393.3\,\text{Hz}$, the natural frequency of mode (2,2). The result using a single control source is shown in Fig. 8(b) with control input $u = 0.7655\sin(2\pi f_4 t - 0.0604)$. Using two control sources as depicted in Fig. 8(c), the control inputs are $u_1 = 0.7579\sin(2\pi f_4 t - 0.0409)$ and $u_2 = 0.1353\sin(2\pi f_4 t - 3.085)$, respectively. The results also show how the number of control sources (1 and 2 control sources) affects the vibration profile of the controlled plate. As expected, better vibration results are obtained with more control sources where the second control input u_2 allows a greater flexibility in modifying the plate's vibration profile.

4.2. Control of vibration profile of a plate

The following numerical study demonstrates the proposed method's ability in modifying the spatial profile of structural vibration. A tonal case is again considered here, which implies modifying the desired spatial vibration profile at a particular frequency ω_o. When a resonance frequency is chosen, the controller attempts to modify the plate's vibration mode shape to be as close as possible to the desired spatial vibration profile $f_{xy}(x,y)$ (see Fig. 9). The spatial weighting $W(x,y)$ is chosen to be unity which means that all regions are

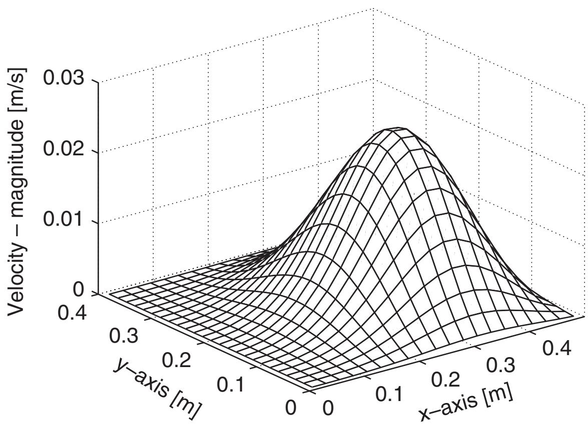

Fig. 9. The reference spatial vibration profile for the plate.

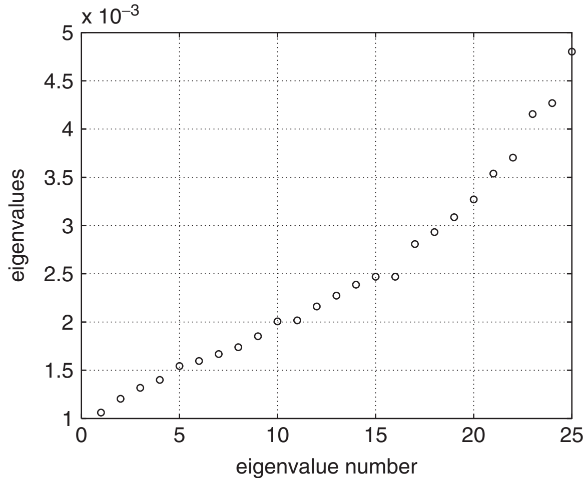

Fig. 10. The eigenvalues plot of $\mathbf{\Omega}_{sp}^{T}\mathbf{\Omega}_{sp}$ for vibration profiling of the plate. Five largest eigenvalues were used for condensation of the spatial filter.

weighted uniformly. The eigenvalues of $\mathbf{\Omega}_{sp}^{T}\mathbf{\Omega}_{sp}$ were analysed (see Fig. 10) and the five largest eigenvalues were used so only five control inputs are generated from the spatial filtering.

Point forces are used for a disturbance source and up to five control sources whose locations are (0.313, 0.222 m) and (0.156, 0.121 m), (0.278, 0.191 m), (0.142, 0.125 m), (0.385, 0.328 m), (0.208, 0.296 m), respectively. The first 3 control sources are used for the first numerical study; and five control sources for the next one. The results for controlling mode (1,1) are shown in Fig. 11. Here, results for three and five control sources are compared, which show that five control sources provide a better vibration profile. Fig. 11(c) shows the spatial error plot for five control sources where the average error over the entire plate is approximately 4%. It was found that more control sources are required to modify the vibration mode shape more accurately, which also depends on the complexity of the desired vibration profile used.

5. Conclusions

A multiple-sensor method for active vibration control of an arbitrary structure has been proposed. This method allows one to spatially control a structure whose dynamic model is not available. The use of spatial

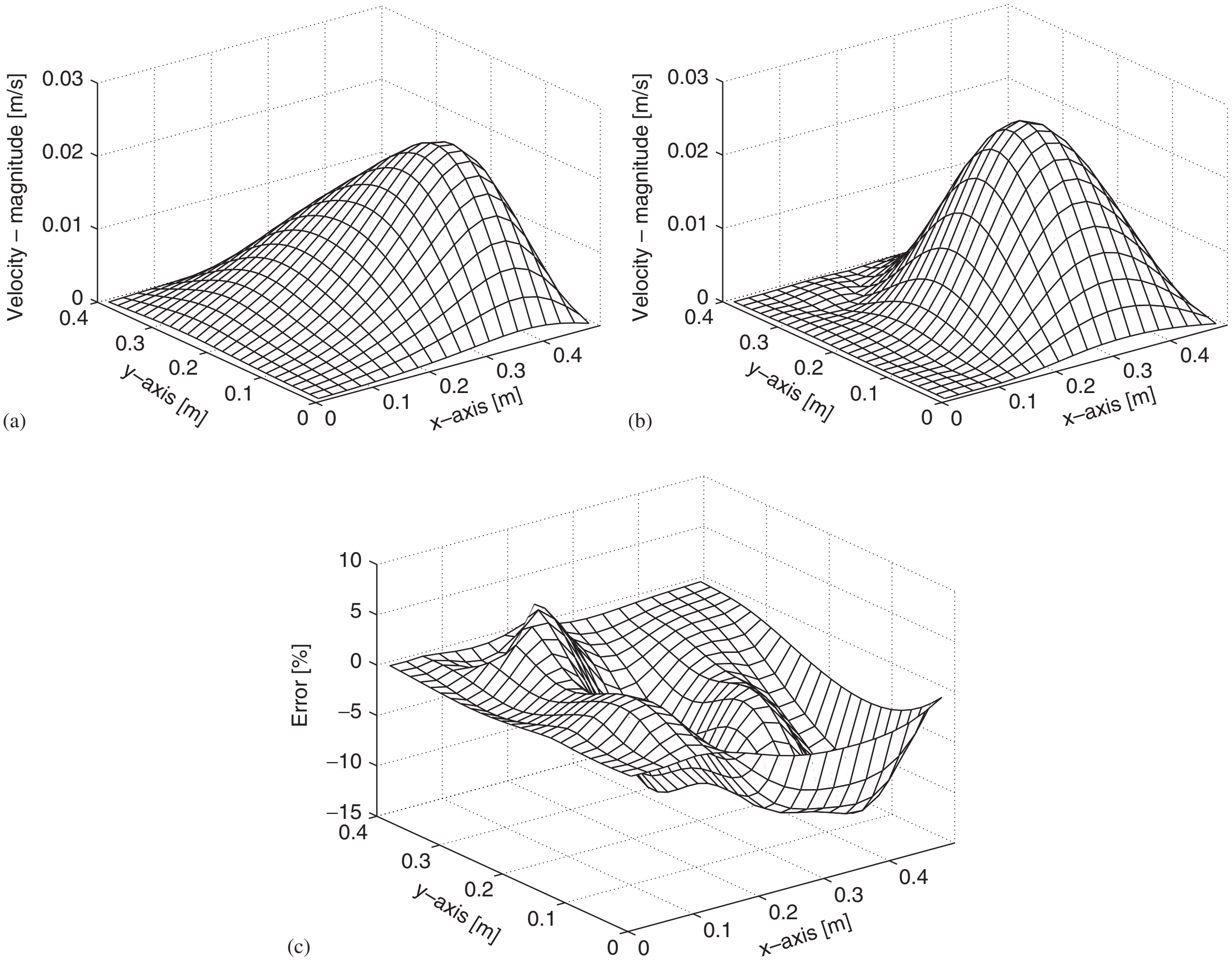

Fig. 11. Comparison of three control and five control sources for modifying the spatial vibration profile of vibration mode (1,1): (a) three control sources; (b) five control sources; (c) spatial error plot for five control sources.

signals are introduced whose signals represent a spatially weighted vibration profile of the structure to be controlled. A case study on a flexible plate demonstrated that the proposed method are capable of controlling structural vibration, either for vibration minimisation or vibration profile modification. One of many applications of such a control method might be the profiling of vibration on flat panel speakers.

Acknowledgements

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